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Algebraical Comparison of Classical and Quantum Polynomial Observables

HANS TILGNER

Institut für theoretische Physik (I) der Universität Marburg, 355 Marburg/Lahn, Renthof 7

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Abstract

The symplectic vector space E of the q and p's of classical mechanics allows a basis free definition of the Poisson bracket in the symmetric algebra over E. Thus the symmetric algebra over E becomes a Lie algebra, which can be compared with the quantum mechanical Weyl algebra with its commutator Lie structure. The universality of the Weyl algebra is used to study the well-known 'classical' Moyal realisation of the Weyl algebra in the symmetric algebra. Quantisations are defined as linear mappings of the underlying vector spaces of the two algebras. It is shown that the classical Lie algebra is –2 graded, whereas the quantum Lie algebra is not. This proves that they are not isomorphic, and hence there is no Dirac quantisation.

1. *Introduction*

In recent years the axiomatic approach to dynamical systems by means of (associative C^*) algebras of observables, initiated by I. Segal, has gained much interest. Independently of this approach the algebraical structure of classical and quantum observables was investigated several times (cf. Groenewold, 1946; Falk, 1953; Uhlhorn, 1957). Since then, abstract algebra has developed new and powerful methods, like basis free descriptions, universality, graduations, etc. They facilitate the algebraical comparison between both dynamics, if one regards only observables, which are polynomials of the canonical position and momentum variables q and p.

In the following both polynomial algebras are constructed over the symplectic vector space (E, σ), spanned by the q's and p's. More general, instead of introducing a symplectic basis $q_1, \ldots, q_n, p^1, \ldots, p^n$ in which the matrix of σ is

$$\begin{pmatrix} 0 & -id_n \\ id_n & 0 \end{pmatrix}$$

only general elements $x \in E$ are considered, and σ is assumed to be a non-degenerate skew bilinear form on E with matrix $\mathscr{I}_\sigma$ only. Thus we avoid trivial alterations of the results such as replacing q_i and p^i by $a_i = q_i + ip^i$ and $a_i^+ = q_i - ip^i$. The ground field is $\mathbb{R}$ or $\mathbb{C}$, abbreviated by $\mathbb{K}$. The classical algebra is realised as the (commutative) symmetric algebra over E. Together with the Poisson bracket it is a -2 graded Lie algebra. The quantum algebra is the Weyl algebra over (E, σ), which is defined by a universal property. This universality allows a unified treatment of the different realisations of the Weyl algebra, like representations (for instance Schrödinger representation in ordinary quantum mechanics) and the well-known Moyal realisation in the symmetric algebra over the dual space of E. The Weyl algebra as a Lie algebra under commutation has no such -2 graduation. This difference of both Lie algebras is used to show that there is no Dirac quantisation (if it is supposed to map the underlying vector spaces onto) and to prove an algebraical version of the Ehrenfest theorem, which states that the time development of a dynamical system coincides in both theories only if the degree of its Hamiltonian does not exceed two.

Notation: We write $ab - ba = [a,b]_-$ and $2[a,b]_+ = ab + ba$ in associative algebras. 'Algebra' means always 'associative algebra with identity element', the latter being written 1 with a typical index. The *directional derivative* of f in the direction $v \in V$ is

$$\Delta^v{}_x f(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{f(x + \varepsilon v) - f(x)\}, \qquad x \in \mathscr{D} \subset V$$

where f is a differentiable function $V \to V'$ of the two vector spaces V, V' and $\mathscr{D}$ is open. $v \mapsto \Delta^v{}_x f(x)$ is a linear mapping $V \to V'$. Hence for every $\mathbb{K}$-valued function f on E there is a uniquely defined $\text{grad}_x f \in E$, called the *gradient* of f with respect to σ, such that

$$\sigma(\text{grad}_x f, v) = \Delta^v{}_x f(x)$$

2. *The Classical Algebra of Polynomial Observables*

Let ten(E) be the tensor algebra over E (Chevalley, 1954) with tensor multiplication $\otimes$. The *symmetric algebra* sym(E) over E is defined by

$$\text{sym}(E) = \text{ten}(E)/(x \otimes y - y \otimes x) \tag{2.1}$$

for $x, y \in E \subset \text{ten}(E)$, where $(x \otimes y - y \otimes x)$ is the two-sided ideal of sym(E) of all elements $X \otimes (x \otimes y - y \otimes x) \otimes Y$ with $X, Y \in \text{sym}(E)$ arbitrary. Obviously ten(E) is commutative and infinite dimensional. If $S^i(E)$ is the $\binom{\dim E + i - 1}{i}$-dimensional vector space generated by the elements of degree i, then

$$\text{sym}(E) = \bigoplus_{i \geqslant 0} S^i(E) \tag{2.2}$$

with $S^0(E) = \mathbb{K}1_S$ and $S^1(E) = E$. If $\wedge$ is the multiplication in sym(E), then

$$S^i(E) \wedge S^k(E) \subset S^{i+k}(E) \tag{2.3}$$

shows that sym(E) is graded.

For $x, w \in E$ the definition $w^\sigma(x) = \sigma(w,x)$ gives a bijection $w \mapsto w^\sigma$ onto the dual space E^σ of E. If one defines

$$(z^\sigma \circ y^\sigma)(x) = z^\sigma(x)\, y^\sigma(x) \tag{2.4}$$

the polynomial algebra generated by E^σ with this multiplication $\circ$ is identical with the symmetric algebra over E^σ and actually the algebra of $\mathbb{K}$-valued polynomials of elements of E with the usual pointwise composition. Its identity will be written 1_{S^*}. For $f, g \in \text{sym}(E^\sigma)$ the *Poisson bracket* $P(f,g)$ of f and g is defined by

$$P(f,g)(x) = \sigma(\text{grad}_x f, \text{grad}_x g) \tag{2.5}$$

(Arnold & Avez, 1967, p. 183). It is skew symmetric and bilinear. A verification, using the chain rule, shows the Jacobi identity. Hence the pair $(\text{sym}(E^\sigma), P)$ is an infinite dimensional Lie algebra. In addition

$$P(f_1 \circ f_2, g) = f_1 \circ P(f_2, g) + P(f_1, g) \circ f_2 \tag{2.6}$$

An easy calculation gives

$$P(1_{S^*}, x^\sigma) = 0, \qquad P(x^\sigma, y^\sigma)(z) = \sigma(x,y) \tag{2.7}$$

$$P(x^\sigma \circ y^\sigma, v^\sigma) = \sigma(x,v)\, y^\sigma + \sigma(y,v)\, x^\sigma \tag{2.8}$$

$$\begin{aligned} P(x^\sigma \circ y^\sigma, v^\sigma \circ z^\sigma) &= \sigma(x,v)\, y^\sigma \circ z^\sigma + \sigma(x,z)\, y^\sigma \circ v^\sigma \\ &\quad + \sigma(y,v)\, x^\sigma \circ z^\sigma + \sigma(y,z)\, x^\sigma \circ v^\sigma \end{aligned} \tag{2.9}$$

More generally, using (2.6) and (2.7), one verifies that the Lie algebra $(\text{sym}(E^\sigma), P)$ is -2 graded, i.e.

$$P(S^i(E^\sigma), S^k(E^\sigma)) \subset S^{i+k-2}(E^\sigma) \tag{2.10}$$

The Lie subalgebra $\mathbb{K}1_{S^*} \oplus E^\sigma$ is the well-known Heisenberg Lie algebra. It is an ideal in the Lie algebra $\mathbb{K}1_{S^*} \oplus E^\sigma \oplus S^2(E^\sigma)$. $S^2(E^\sigma)$ will be identified below.

The isomorphism $w \mapsto w^\sigma$ of E and E^σ can be used to transport the Poisson bracket and all the algebraical results from sym(E^σ) to sym(E).

It can be shown by a method, given by Wollenberg (1969a) in the symplectic basis, that the Poisson bracket is not a commutator of an associative algebra multiplication on sym(E^σ).

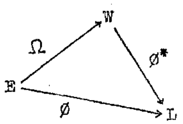
3. *The Quantum Algebra of Polynomial Observables*

We want to define a universal algebra in which (E,σ) is embedded in such a way that the commutator of $x, y \in E$ is identical to $\sigma(x,y)$.

More exactly: A *Weyl algebra* over (E,σ), written weyl(E,σ), is a pair (W,Ω) of an algebra W and a linear mapping $\Omega: E \to W$ such that for every linear mapping ϕ of E into an algebra L with

$$\phi(x)\phi(y) - \phi(y)\phi(x) = æ(x,y)\,1_L, \qquad æ(x,y) \in \mathbb{K} \tag{3.1}$$

there is a unique algebra homomorphism $\phi^*: W \to L$ such that the diagram



is commutative, i.e. $\phi^* \circ \Omega = \phi$ (Nouzae & Revoy, 1971, p. 7).

Theorem: (a) Given two Weyl algebras (W,Ω) and (W',Ω') of (E, σ), there is a unique isomorphism Ω'': $W \to W'$ with $\Omega'' \circ \Omega = \Omega'$. (b) W is generated by the image $\Omega(E)$ and its identity 1_W. (3.2)

The proof is standard (Lang, 1965, p. 367; Jacobson, 1961, p. 153). In the following the construction of a Weyl algebra is given. Denote by Γ the restriction to $E \subset$ ten(E) of the canonical projection π from ten(E) onto the algebra

$$\text{ten}(E)/(x \otimes y - y \otimes x - \sigma(x,y)\,1_t) \tag{3.3}$$

for $x, y \in E \subset$ ten(E). Following Jacobson (1961), p. 155, one proves

Theorem: The pair $(\text{ten}(E)/(x \otimes y - y \otimes x - \sigma(x,y)\,1_t),\ \Gamma)$ is a Weyl algebra over (E,σ). (3.4)

The algebra (3.3) was already defined by Segal (1968, p. 148). Trivially $\pi(x \otimes y - y \otimes x - \sigma(x,y)\,1_t) = 0$, and (3.2a) shows that (3.1) is valid in any Weyl algebra. Using this and (3.2) one proves that for every element D of the symplectic matrix Lie algebra der(E,σ), there is a unique derivation D^* of W with $\Omega \circ D = D^* \circ \Omega$ (Jacobson, 1961, p. 154). The realisation (3.3) of a Weyl algebra shows that E is embedded injectively and that its dimension is infinite. Nouaze & Revoy have shown that the centre consists of multiples of the identity only (1971, p. 17), that it is simple (1971, p. 24) and that all derivations are inner (1971, p. 29).

The Weyl algebra is the analog of the Clifford algebra of a (pseudo-) orthogonal vector space, if one substitutes σ for the symmetric bilinear form of the latter (Chevalley, 1954, p. 33; Land, 1965, p. 367). For vanishing bilinear form they reduce to the symmetric and the exterior algebra respectively.

For $x, y \in E \subset$ weyl(E,σ) one deduces the *canonical commutation* relations

$$xy - yx = \sigma(x,y)\,1_W \tag{3.5}$$

from (3.3). Thus the vector space $\mathbb{K}1_W \oplus E$ together with the commutator is a Heisenberg Lie algebra. With the help of this, we can give another useful characterisation of the Weyl algebra. Given its universal enveloping algebra $ue(\mathbb{K}1_W \oplus E)$, 1_W and 1_{ue} are different from each other (Jacobson, 1961, p. 160). Denote by Ω' the embedding of $\mathbb{K}1_W \oplus E$ into $ue(\mathbb{K}1_W \oplus E)$, by π the canonical projection of $ue(\mathbb{K}1_W \oplus E)$ onto

$$ue(\mathbb{K}1_W \oplus E)/(1_W - 1_{ue}) \tag{3.6}$$

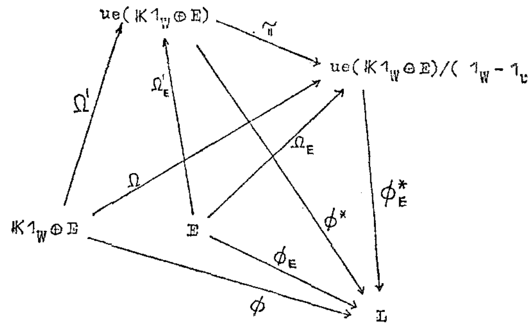
and by Ω_E the (injective) restriction of $\Omega = \pi \circ \Omega'$ to E.

Theorem: The pair $(ue(\mathbb{K}1_W \oplus E)/(1_W - 1_{ue}), \Omega_E)$ is a Weyl algebra over (E, σ). (3.7)

Proof: From its construction, Ω_E has the property (3.1) with $æ = \sigma$. Every such mapping gives rise to a unique Lie algebra homomorphism ϕ: $\mathbb{K}1_W \oplus E \to L$. Hence from the definition of the universal enveloping algebra there is a unique homomorphism ϕ^*: $ue(\mathbb{K}1_W \oplus E) \to L$ with $\phi = \phi^* \circ \Omega'$. Let us define a homomorphism

$$\phi_E^*: ue(\mathbb{K}1_W \oplus E)/(1_W - 1_{ue}) \to L$$

by $\phi = \phi_E^* \circ \pi$.



It is unique and $\phi = \phi_E^* \circ \pi \circ \Omega' = \phi_E^* \circ \Omega$. Thus restriction to E gives $\phi_E = \phi_E^* \circ \Omega_E$, which proves the universality. ∎

By means of the canonical projection π one can transport results from the universal enveloping algebras to the Weyl algebras; for instance the Poincaré-Birkhoff-Witt theorem (Jacobson, 1961, p. 159) is valid in an analogous form. However, for the comparison with the classical algebra it is more convenient to study another basis, which is given by the totally symmetrised monomials of a basis of E and 1_W (Helgason, 1962, p. 392). If γ_r denotes the symmetric permutation group of r objects, we write for $x_i \in E$

$$\Lambda x_1 x_2 \dots x_r = \frac{1}{r!} \sum_{\tau \in \gamma_r} x_{\tau(1)} x_{\tau(2)} \dots x_{\tau(r)} \tag{3.8}$$

The vector space ΛW_r spanned by the symmetrised monomials (3.8) of degree r is $\binom{\dim E + r - 1}{r}$-dimensional (Tilgner, 1970, p. 118). Writing

$\Lambda W_0 = \mathbb{K}1_W$ and $\Lambda W_1 = E$, one has

$$\mathrm{weyl}(E,\sigma) = \bigoplus_{i\geq 0} \Lambda W_i \tag{3.9}$$

$$[\Lambda x_1 \ldots x_r, y]_- = \Lambda[x_1 \ldots x_r, y]_-, \qquad x_i, y \in E \tag{3.10}$$

i.e. $[\Lambda W_r, E]_- \subset \Lambda W_{r-1}$, and for $u, v \in E$

$$[\Lambda uv, \Lambda x_1 \ldots x_r]_- = \Lambda[uv, x_1 \ldots x_r]_- \tag{3.11}$$

i.e. $[\Lambda W_2, \Lambda W_r]_- \subset \Lambda W_r$ (Tilgner, 1970). However, this can be generalised for $i,k > 2$ only in the form

$$[\Lambda W_i, \Lambda W_k]_- \subset \bigoplus_{m\geq 1} \Lambda W_{i+k-2m} \tag{3.12}$$

where one cannot drop the summation over m, as shows the following example. Using $\Lambda xy = xy - \frac{1}{2}\sigma(x,y)1_W$ and

$$\Lambda xyz = xyz + \tfrac{1}{2}\{-\sigma(x,y)z + \sigma(z,x)y - \sigma(y,z)x\} \tag{3.13}$$

$$\begin{aligned}\Lambda xyzw = xyzw &- \tfrac{1}{2}\{\sigma(x,y)\Lambda zw + \sigma(x,z)\Lambda yz + \sigma(x,w)\Lambda yz \\ &+ \sigma(y,z)\Lambda xw + \sigma(y,w)\Lambda xz + \sigma(z,w)\Lambda xy\} \\ &- \tfrac{1}{4}\{\sigma(x,y)\sigma(z,w) + \sigma(x,z)\sigma(y,w) + \sigma(x,w)\sigma(y,z)\}1_W\end{aligned} \tag{3.14}$$

an easy but tedious calculation gives

$$\begin{aligned}[\Lambda xyz, \Lambda uvw]_- = \Lambda[xyz, uvw]_- &+ \tfrac{1}{4}\{\sigma(x,u)\sigma(y,v)\sigma(z,w) \\ &+ \sigma(x,u)\sigma(y,w)\sigma(z,v) + \sigma(x,v)\sigma(y,w)\sigma(z,u) \\ &+ \sigma(x,v)\sigma(y,u)\sigma(z,w) + \sigma(x,w)\sigma(y,u)\sigma(z,v) \\ &+ \sigma(x,w)\sigma(y,v)\sigma(z,u)\}1_W\end{aligned} \tag{3.15}$$

i.e. $[\Lambda W_3, \Lambda W_3]_- \subset \Lambda W_4 \oplus \mathbb{K}1_W$. This shows that the Lie algebra $(\mathrm{weyl}(E,\sigma), [,]_-)$ is not -2 graded in the decomposition (3.9).

4. *Quantisations*

The problem of quantisation has been treated often in physics (cf. Ahrens & Babitt, 1965; Souriau, 1966; Streater, 1966; Agarwal & Wolf, 1970; Shankara & Srinivas, 1971 and references therein).

Definition: A *quantisation* is a vector space isomorphism Ψ of $\mathrm{sym}(E)$ onto $\mathrm{weyl}(E,\sigma)$ with $\Psi 1_S = 1_W$. If in addition Ψ is an isomorphism of the Lie algebras $(\mathrm{sym}(E), P)$ and $(\mathrm{weyl}(E,\sigma), [,]_-)$ then Ψ is a *Dirac quantisation.* (4.1)

Comparison of (3.17) and (2.10) shows that the quantisation Λ': $\mathrm{sym}(E) \to \mathrm{weyl}(E,\sigma)$, defined by

$$\Lambda' : x_1 \wedge \ldots \wedge x_r \mapsto \Lambda x_1 \ldots x_r, \qquad x_i \in E \tag{4.2}$$

is no Dirac quantisation. It should be called *Weyl quantisation* since it is exactly the Weyl rule of associating a quantum mechanical observable to a given classical polynomial (Agarwal & Wolf, 1970, p. 2179). Besides the symmetrised basis of weyl(E,σ) there is another class of bases due to the Poincaré-Birkhoff-Witt theorem. Those bases consist of 1_W and the monomials of basis elements of E in some fixed order, 'standard monomials' (Jacobson, 1961, p. 156). Each of them allows a direct decomposition of type (3.9) of weyl(E,σ) into vector spaces of standard monomials of equal degree, and defines a quantisation. However, comparison of (2.9) with

$$\begin{aligned}[xy, vz]_- &= \sigma(y,z)\,xv + \sigma(y,v)\,xz + \sigma(x,z)\,vy + \sigma(x,v)\,zy \\ &= \sigma(y,z)\,xv + \sigma(y,v)\,xz + \sigma(x,z)\,yv + \sigma(x,v)\,yz \\ &\quad - \{\sigma(x,z)\,\sigma(y,v) + \sigma(x,v)\,\sigma(y,z)\}\,1_W \end{aligned} \tag{4.3}$$

shows, that none of those bases induces a Dirac quantisation.

More general: There is no Dirac quantisation: From $P(\Psi^{-1}x, \Psi^{-1}y) = \sigma(x,y)\,1_S$ follows that $\Psi^{-1}E = E$. In addition $\sigma(\Psi x, \Psi y)\,1_W = [\Psi x, \Psi y]_- = \Psi P(x,y) = \sigma(x,y)\,1_W$ (hence $\Psi|_E$ is in the symplectic matrix group). Next from $P(\Psi^{-1}\Lambda W_2, \Psi^{-1}E) = \Psi^{-1}[\Lambda W_2, E]_- \subset \Psi^{-1}E$ and (2.8) follows $\Psi S^2(E) = \Lambda W_2$; continuation gives, $\Psi S^r(E) = \Lambda W_r$ for all r. But then $[\Psi S^3(E), \Psi S^3(E)]_- = \Psi P(S^3(E), S^3(E)) \subset \Lambda W_4$ is a contradiction to (3.17).

5. *The Moyal Realisation of the Weyl Algebra*

Given two $\mathbb{K}$-valued functions on E, say $f(z)$ and $g(w)$, we write

$$\sigma(\partial_z, \partial_w) f(z) g(w) = \sigma(\mathrm{grad}_z f, \mathrm{grad}_w g) \tag{5.1}$$

Inserting a basis of E, one can show that $\sigma(\partial_z, \partial_w)$ is a differential operator. Thus it can be iterated. For all $f, g \in \mathrm{sym}(E^\sigma)$, $M_æ(f,g)$ defined by

$$M_æ(f,g)(x) = [\exp æ\sigma(\partial_z, \partial_w)] f(z) g(w)|_{z=w=x}, \qquad æ \in \mathbb{K} \tag{5.2}$$

is an element of sym(E^σ), because the exponential series breakes. $(f,g) \mapsto M_æ(f,g)$ is a bilinear composition on sym(E^σ) and 1_{S^*} is the identity with respect to it. It is well known (Grossmann *et al.*, 1968) that $M_æ(f,g)$ is associative; their proof can be made basis free immediately. A verification gives

$$M_æ(x^\sigma, y^\sigma) = x^\sigma \circ y^\sigma + æ\sigma(x,y)\,1_{S^*} \in S^2(E^\sigma) \oplus \mathbb{K}1_{S^*} \tag{5.3}$$

$$\begin{aligned} M_æ(M_æ(x^\sigma, y^\sigma), z^\sigma) &= x^\sigma \circ y^\sigma \circ z^\sigma + æ\{\sigma(x,y)\,z^\sigma - \sigma(z,x)\,y^\sigma \\ &\quad + \sigma(y,z)\,x^\sigma\} \in S^3(E^\sigma) \oplus E^\sigma \end{aligned} \tag{5.4}$$

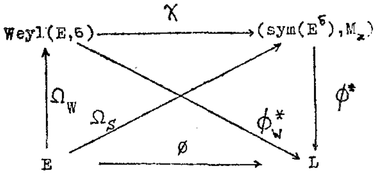
cf. (3.13). From (5.3)

$$M_æ(x^\sigma, y^\sigma) - M_æ(y^\sigma, x^\sigma) = 2æ\sigma(x,y)\,1_{S^*} \tag{5.5}$$

i.e. $\mathbb{K}1_{S^*} \oplus E^\sigma$ is the familiar Heisenberg Lie algebra with respect to the commutator of M.

Theorem: The associative algebra $(\mathrm{sym}(E^\sigma), M_\ae)$ is a Weyl algebra over (E,σ). (5.6)

Proof: For (injective) embedding of E into $\mathrm{sym}(E^\sigma)$ we take Ω_S: $E \to E^\sigma$ $\Omega_S: x \mapsto x^\sigma$.



From (5.5) and the definition of the Weyl algebra there are unique homomorphisms χ: $\mathrm{weyl}(E,\sigma) \to (\mathrm{sym}(E^\sigma), M_\ae)$ with $\Omega_S = \chi \circ \Omega_W$ and $\phi_W{}^*$: $\mathrm{weyl}(E,\sigma) \to L$ with $\phi = \phi_W{}^* \circ \Omega_W$, if ϕ fulfills (3.1). Define a homomorphism ϕ^*: $(\mathrm{sym}(E^\sigma), M_\ae) \to L$ by $\phi_W{}^* = \phi_W \circ \chi$. It is unique and $\phi = \phi^* \circ \chi \circ \Omega_W = \phi^* \circ \Omega_S$ which proves the universality.

Corollary: $\mathrm{sym}(E^\sigma)$ is generated by E^σ with respect to the multiplication $M_\ae$ and $(\mathrm{sym}(E^\sigma), M_\ae)$ is isomorphic to the algebra (3.3). (5.7)

However, this establishes no solution of the Dirac quantisation problem since the commutator with respect to $M_\ae$ is not the Poisson bracket. For $\ae = \frac{1}{2}i\hbar$ (5.5) are the canonical commutation relations of quantum mechanics in the Moyal phase space formulation. In this case the commutator becomes

$$i[\sin \tfrac{1}{2}\hbar\sigma(\partial_z, \partial_w)] f(z) g(w)|_{z=w=x} \tag{5.8}$$

which is the Moyal bracket (Moyal, 1949). For $\hbar \to 0$ it reduces to the Poisson bracket. That it is a Lie bracket was first shown by Jordan & Sudarshan (1961). For $\mathbb{K} = \mathbb{R}$ only $\ae = \frac{1}{2}$ is possible and in (5.8) one has the sinh series.

6. *The Polynomials of Second Degree*

The vector spaces $S^2(E)$ and ΛW_2 are Lie subalgebras of $(\mathrm{sym}(E), P)$ and $(\mathrm{weyl}(E,\sigma), [,]_-)$ respectively. $(\Lambda W_2, [,]_-)$ is isomorphic to the symplectic matrix Lie algebra in (E,σ) (Tilgner, 1971) .This proof can be applied to $(S^2(E), P)$ too.

More general: From the Jacobi identity and (2.10) follows that the vector spaces $S^i(E)$ are an infinite series of representation spaces for $(S^2(E), P)$. The corresponding facts hold in $\mathrm{weyl}(E,\sigma)$ (Tilgner, 1970, p. 126).

Following Uhlhorn (1957, p. 97), an algebraic version of the Ehrenfest theorem is proved, giving a second application of (3.17). For this let us recall some facts on time developments of observables in a purely algebraical framework. A *state* on the algebra of observables is a (positive) linear functional ($\mathbb{K}$-valued). The time development of a dynamical system is given by a one-parameter subgroup of the automorphism group of the algebra. Since $\mathrm{weyl}(E,\sigma)$ is simple (Nouaze & Revoy, 1971, p, 24) every derivation is inner (ibid., p. 29). Thus the time development of $X \in \mathrm{weyl}(E,\sigma)$ is given by $X(\tau) = (\exp \tau \,\mathrm{ad}(H))X$, $\tau \in \mathbb{R}$, for some $H \in \mathrm{weyl}(E,\sigma)$, called the *Hamiltonian operator*, $X(0) = X$. Note that the exponential series breakes. The time development of the expectation value $\omega(X)$ in the state ω is given by $\omega(X) \mapsto \omega_\tau(X) := \omega(X(\tau))$. One easily proves that

$$\Delta_\tau^1 \omega_\tau(X) = \omega([H, X]_-)$$

The condition that the classical and quantum time developments coincide is

$$\omega(M_\text{æ}(H, X) - M_\text{æ}(X, H)) = \omega(P(H, X))$$

with H, $X \in \mathrm{sym}(E^\sigma)$. Thus for all $X \in \mathrm{weyl}(E,\sigma)$, the time developments coincide if the degree of H does not exceed two.

Note that Wollenberg (1969b) has proved, that the classical Lie algebra $(\mathrm{sym}(E^\sigma), P)$ has outer derivations.

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